American Mathematical Society

Transactions of the American Mathematical Society

Edited by D. R. CURTISS, L. P. EISENHART, P. F. SMITH, with the co-operation of G. D. BIRKHOFF, A. B. COBLE, W. A. HURWITZ, DUNHAM JACKSON, EDWARD KASNER, W. R. LONGLEY, C. N. MOORE, R. L. MOORE, F. R. MOULTON, ARTHUR RANUM, F. R. SHARPE, E. J. WILCZYNSKI.

The *Transactions* is devoted primarily to research in pure and applied mathematics, and is the official organ of the Society for the publication of important original papers read before it.

Published quarterly by the Society. Subscription price for the annual volume, $5.00; to members of the Society, $3.75. Subscriptions should be sent to J. H. TANNER, Treasurer of the American Mathematical Society, Cayuga Heights, Ithaca, N. Y.

Bulletin of American Mathematical Society

Published monthly, except August and September. Subscription price for the annual volume, $5.00. A member of the Society connected with an educational institution may order the current volume for its library at $3.00. So far as the supply on hand permits, members are entitled to receive back volumes of the Bulletin to complete their personal sets at $4.00 per volume. Vols. I-X are out of print. Orders and subscriptions should be addressed to the Society, 501 West 116th Street, New York City.

American Mathematical Society

Mathematical Papers read at the International Mathematical Congress

held in connection with the World's Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress: E. H. Moore, Oskar Bolza, Heinrich Maschke, H. S. White.

Published for the Society, 1896, by The Macmillan Company, New York, N. Y. Orders should be addressed to the Society, 501 West 116th Street, New York City. Price, $3.00; to members of the Society, $1.50.

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Published, 1914, by the Society, to which orders should be addressed. Price, $2.00; to members of the Society, $1.50.

BULLETIN

OF THE

AMERICAN

MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

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VOLUME XXIII., NUMBER 7

APRIL, 1917

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1917

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 257 $5.00 a year

The American Mathematical Monthly

OFFICIAL JOURNAL OF

The Mathematical Association of America

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-ninth regular meeting of the Society was held in New York City on Saturday, February 24. The morning session sufficed for the presentation of the brief list of papers. The attendance included the following twenty-six members:

Mr. D. R. Belcher, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor C. C. Grove, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. Harry Langman, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor W. B. Stone, Mr. H. E. Webb, Professor H. S. White, Mr. J. K. Whittemore.

Professor H. S. White occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor H. P. Kean, McHenry College; Mr. Ralph Keffer, Harvard University; Mr. H. C. M. Morse, Harvard University; Dr. F. D. Murnaghan, Rice Institute; Mr. G. E. Raynor, University of Washington; Dr. S. P. Shugert, University of Pennsylvania; Mr. G. W. Smith, University of Illinois; Mr. J. S. Taylor, University of California; Dr. L. E. Wear, University of Washington; Dr. H. N. Wright, University of California. Four applications for membership in the Society were received.

It was decided to hold the next summer meeting of the Society at Cleveland, Ohio, on September 4–5. The Mathematical Association of America will meet at Cleveland on September 6–7. The two organizations have appointed a joint committee on arrangements for the meetings, consisting of Professors Focke, Huntington, Pitcher, D. T. Wilson, and Secretaries Cole and Cairns.

At the annual meeting of the Society, the Council placed itself on record as desiring to cooperate with the National Research Council in forwarding the interests of research. At the February meeting, a committee was appointed to confer with

the chairman of the Mathematics Committee of the Research Council, Professor E. H. Moore, in regard to the selection of the members of that committee.

By the will of the late Professor L. L. Conant, who was a member of the Society from 1892 to 1916, the sum of $10,000 is left to the Society, subject to Mrs. Conant's life interest. The will provides that the income of this bequest "shall be offered once in five years as a prize for original work in pure mathematics." This generous gift, a noble monument to the donor, should do much for the promotion of higher mathematical aims in this country. For many years the Society has consistently pursued these aims, with a success far outrunning what might have been expected from its modest financial resources. With greater means in the way of general or special funds, it could accomplish still more. To anyone who is able to give for science, the Society presents itself as an experienced and beneficent administrator.

The following papers were read at the February meeting:

(1) Dr. A. R. SCHWEITZER: "The iterative compositions of a function of $n+1$ variables ($n = 1, 2, 3, \cdots$)."

(2) Dr. A. R. SCHWEITZER: "Functional equations based on iterative compositions."

(3) Dr. D. F. BARROW: "An application of Fourier's series to probability."

(4) Professor EDWARD KASNER: "Degenerate cases in the theory of conduction of heat."

(5) Professor J. E. ROWE: "The equation of a rational plane curve derived from its parametric equations (second paper)."

(6) Professor R. L. MOORE and Dr. J. R. KLINE: "The most general closed plane point set through which it is possible to pass a simple continuous plane arc."

(7) Professor H. S. WHITE: "New proof of a theorem of von Staudt and Hurwitz."

(8) Professor HENRY TABER: "On the structure of finite continuous groups."

In the absence of the authors, the papers of Dr. Schweitzer, Dr. Barrow, Professor Rowe, and Professor Taber were read by title. Abstracts of the papers follow below.

1. Dr. Schweitzer constructs an exhaustive table of iterative compositions of a function of $n+1$ variables ($n = 1, 2, 3, \cdots$). A symbolic representation for such a composition

is used, and by introducing a simple convention the compositions are readily ordered for each value of n. The construction of the table is based on the general principle implied by the following operation: If the function $f(x_0, x_1, \cdots, x_n)$ has some one of its arguments, say x_0, replaced by the function f, then this f is denoted by $f(x_{00}, x_{01}, \cdots, x_{0n})$ and the resulting composition is denoted by the symbol

$$[01, \cdots, n; (00, 01, \cdots, 0n), 1, \cdots, n].$$

More compactly, the author puts, e. g., $F(01) \equiv f(x_0, x_1)$, $F(01; 1) \equiv [F(01); 0, (10, 11)]$, etc. Similarly for three variables one obtains $F(012)$, $F(012; 1)$, etc., $F(012; 11)$, etc. Any iterative composition of a function of $n+1$ variables is represented, then, by the symbol $F(012 \cdots n; i_1, i_2, \cdots)$ where i_1, i_2, etc., are certain integers. To illustrate, for $n = 1$ the iterative compositions are $f(x_0, x_1)$, $f\{x_0, f(x_{10}, x_{11})\}$, $f\{f(x_{00}, x_{01}), x_1\}$, $f\{f(x_{00}, x_{01}), f(x_{10}, x_{11})\}$, $f\{x_0 f[x_{10}, f(x_{110}, x_{111})]\}$, etc.

2. In Dr. Schweitzer's second paper a systematic application is made of the preceding iterative compositions of a function of $n+1$ variables ($n = 1, 2, 3, \cdots$) to the genesis of functional equations, by means of the inversion, elimination, or iteration of some or all of the involved variables (the processes being taken separately or in combination). The author obtains three categories of functional equations, viz., the inversive, the eliminative, and the iterative, which, however, are not necessarily mutually exclusive. It is found that certain eliminative functional equations, including certain of those called "quasi-transitive," are implied by certain purely inversive functional relations. An important class of inversive functional equations is generated by the following requirement: Given the iterative composition $F(012 \cdots n; i_1, i_2, \cdots)$ and a substitution group G on the variables involved (leaving some or none of them fixed); to find the function $f\{x_1, x_2, \cdots, x_{n+1}\}$ which satisfies the functional equations expressed by the formal invariance of the composition $F(012 \cdots n; i_1, i_2, \cdots)$ under the substitutions of the group G. On the other hand extensive new categories of eliminative functional equations are obtained. A few of the theorems proved are as follows:

I. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the cyclic group on x_1, x_2, x_3, x_4 then $f(x_1, x_2) = x^{-1}\{cx(x_1) + cx(x_2)\}$ where c is an arbitrary constant.

II. This theorem is a direct generalization of theorem I and incidentally generalizes Hayashi's generalization of Abel's functional equations. Instead of the symmetric group the cyclic group is used.

III. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the non-cyclic group of order four on x_1, x_2, x_3, x_4, and if $f(x, x) = \text{const.}, f\{f(x_1, x_2)c_1\} = \lambda f(x_1, x_2)$, then $f(x_1, x_2) = x^{-1}\{cx(x_1) - cx(x_2)\}$, where c is an arbitrary constant.

IV. If $f\{x, f(t_1, t_2, y), f(t_1, t_2, z)\} = f(x, y, z)$, then $f(x, y, z) = x^{-1}\{\vartheta(x) - x(y) + x(z)\}$, where $\vartheta(x)$ is arbitrary.

V. If $f\{f(x, y), f[x, f(x, \cdots, f\{x, z\})]\} = f(z, y)$, where in the latter equation the symbol f occurs $n + 3$ times $(n = 1, 2, 3, \cdots)$, then $f(x, y) = x^{-1}\{x(x) + wx(y)\}$, where w is a root of the equation $w^n + w^{n-1} + \cdots + w + 1 = 0$.

3. This paper is the outgrowth of an attempt on the part of Dr. Barrow to solve a problem proposed by Professor E. W. Brown. A solution of the problem and a new proof of certain well known properties of frequency curves are obtained as applications of the main theorem of the paper which is as follows:

Let two quantities N and M have frequencies $f(x)$ and $\varphi(x)$ respectively, and let $F(x)$ denote the frequency of $M + N$. If these three functions are identically zero outside some interval $(-C, C)$, and capable of development in Fourier's series within this interval, then, letting a_i and b_i, α_i and β_i, A_i and B_i denote the ith coefficients of the cosines and sines in these series, we have

$$A_i + B_i\sqrt{-1} = C(a_i + b_i\sqrt{-1})(\alpha_i + \beta_i\sqrt{-1}).$$

Equating the real and imaginary parts enables us to calculate the coefficients in the development of $F(x)$ from those in the developments of $f(x)$ and $\varphi(x)$.

4. The partial differential equation for the conduction of heat in a plane determines the temperature as a function of the position (x, y) and the time t. For each solution $\varphi(x, y, t)$ there will be, in general, ∞^2 isothermal curves $\varphi(x, y, t) = c$.

Professor Kasner finds all solutions for which the doubly infinite system degenerates into a simply infinite system. There are three types: In the first $\varphi = f(x, y)$, the temperature not entering; in the second, $\varphi = f(x, y) + t$; in the third, $\varphi = e^t f(x, y)$. The differential equations for these types are respectively $\Delta f = 0$, $\Delta f = 1$, $\Delta f = f$, where $\Delta f \equiv f_{xx} + f_{yy}$.

5. Professor Rowe's paper appears in full in the present number of the BULLETIN.

6. In order that it may be possible to pass a simple continuous arc through a given closed point set M it is sufficient that M should contain no connected subset consisting of more than one point. Cf. F. Riesz, *Comptes Rendus*, volume 141 (1905), page 650, A. Denjoy, ibid., volume 151 (1910), page 140, and L. Zoretti, ibid., volume 142 (1906), page 763. It is clear that this is not a necessary condition. Professor Moore and Dr. Kline have obtained the following theorem:

In order that it may be possible to pass at least one simple continuous arc through a given closed point set M it is necessary and sufficient that every closed connected subset of M should be either a single point or a simple continuous arc k no point of which, except its end points, is a limit point of $M - k$.

7. The theorem concerning two tetrahedra inscribed in a twisted cubic curve was announced by von Staudt in his Beiträge, and proved by Hurwitz thirty years later. Professor White points out that a simpler proof is found by projecting the curve, from all its points, into a plane. Then the theorem in question is derived easily from the well-known situation of two triangles inscribed to a conic, which is the Poncelet theorem for triangles.

8. In Professor Taber's paper $X_1, \cdots, X_r$ are the infinitesimal transformations of a finite continuous group G, in which case $(X_i, X_j) = \Sigma_1^r c_{ijk} X_k$ $(i, j = 1, 2, \cdots, r)$. Any infinitesimal transformation $\Sigma \alpha_i X_i$ of G can be represented by the point $\alpha = (\alpha_1, \cdots, \alpha_r)$ in the space of $r - 1$ dimensions upon which the group Γ adjoint to G operates. If now $f(\alpha) \equiv f(\alpha_1, \cdots, \alpha_r)$ is an invariant of Γ, it follows that the alternant of $\Sigma \alpha_i X_i$ and any other infinitesimal transformation of G is represented by

a point which lies in the polar $r-2$ flat of α qua $f(\alpha)=0$. Whence it follows that any $r-2$ flat $\Sigma u_i\alpha_i=0$ is invariant to each of the infinitesimal transformations of Γ represented by the poles of this flat qua $f(\alpha)=0$; and thus, if the poles of this flat do not all lie in an $r-2$ flat, it follows that the infinitesimal transformations of G represented by points in the $r-2$ flat $\Sigma u_i\alpha_i=0$ generate an invariant subgroup of G.

If the adjoint group Γ has a quadratic invariant of non-zero discriminant, for a proper choice of the X's, we shall have $c_{ijk}+c_{ikj}=0$ $(i, j, k=1, 2, \cdots, r)$. In this case the condition that $f(\alpha)$ shall be invariant to Γ is that

$$\Sigma\alpha_i E_i\left(\frac{\partial f(\alpha)}{\partial\alpha_1}, \cdots, \frac{\partial f(\alpha)}{\partial\alpha_r}\right)=0,$$

where E_i is the matrix whose constituent in the μth row and νth column is $c_{i\nu\mu}$ $(i, \mu, \nu=1, 2, \cdots, r)$; or, what is the same thing, is that

$$\left(\Sigma\alpha_i X_i, \Sigma\frac{\partial f(\alpha)}{\partial\alpha_i}X_i\right)=0$$

for all values of the α's. Whence it follows, if $f(\alpha)$ is a second invariant of Γ, that the infinitesimal transformation $\Sigma u_i X_i$ is commutative with every infinitesimal transformation of G represented by a pole, qua $f(\alpha)=0$, of the $r-2$ flat $\Sigma u_i\alpha_i=0$; and, if these poles do not all lie in any $r-2$ flat, it follows that $\Sigma u_i X_i$ is an exceptional infinitesimal transformation.

F. N. COLE,
Secretary.

THE EQUATION OF A RATIONAL PLANE CURVE DERIVED FROM ITS PARAMETRIC EQUATIONS (SECOND PAPER).

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society, February 24, 1917.)

As this is the second article on the same subject published by the author in the BULLETIN, it is desirable to inform the reader at once that the method of deriving the equation of a rational plane curve from its parametric equations described in this paper is published not merely because it is a new

method but also because in many respects it is an improvement upon previous methods. To verify this statement the reader is referred to the processes given by Salmon (Higher Plane Curves, § 44), by Grace and Young (Algebra of Invariants, § 254), and to the method of my former article,* which was also developed independently by Mr. H. W. Richmond, of Cambridge University, but not published by him until after my article had appeared.

As in the preceding article, it seems best for the sake of readability to present the method through the instrumentality of the easiest illustrations.

First, let

$$\rho x_i = a_i t^2 + b_i t + c_i \qquad (i = 0, 1, 2) \tag{1}$$

be the parametric equations of a conic or R^2. By multiplying† each of these equations by k, transposing all terms to the same side, and replacing $-k\rho x_i$ by σx_i, we obtain

$$a_i kt^2 + b_i kt + c_i k + \sigma x_i = 0 \qquad (i = 0, 1, 2). \tag{2}$$

Solving (2) for the ratios of kt^2, kt, k, σ regarded as independent quantities, we have

$$kt^2 : -kt : k : -\sigma = |bcx| \ddagger : |acx| : |abx| : |abc|. \tag{3}$$

From (3)

$$-t = \frac{kt^2}{-kt} = \frac{-kt}{k} = \frac{|bcx|}{|acx|} = \frac{|acx|}{|abx|}. \tag{4}$$

Hence from (4)

$$\begin{aligned} |abx|\,t + |acx| &= 0, \\ |acx|\,t + |bcx| &= 0. \end{aligned} \tag{5}$$

Eliminating t we have

$$\begin{vmatrix} |abx| & |acx| \\ |acx| & |bcx| \end{vmatrix} = 0,$$

and this is the equation of the R^2 of (1). Obviously in this

* This BULLETIN, vol. 22, No. 7 (April, 1916), pp. 338–340. Also, see Professor Richmond's article, vol. 23, No. 2 (November, 1916), pp. 90–91.

† This is merely a matter of personal preference.

‡ By $|abx|$ is meant the three-rowed determinant $\begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}$.

simple case the same result might have been obtained by simplifying the last equality in (4).

Further, let

$$\rho x_i = a_i t^3 + b_i t^2 + c_i t + d_i \qquad (i = 0, 1, 2) \tag{6}$$

be the parametric equations of the rational plane cubic R^3. Following (2), these may be written

$$a_i k t^3 + b_i k t^2 + c_i k t + d_i k + \sigma x_i = 0 \quad (i = 0, 1, 2). \tag{7}$$

These may be arranged in a form similar to (2) in a number of ways; for example,

$$a_i t \cdot k t^2 + b_i t \cdot k t + (c_i t + d_i) k + \sigma x_i = 0, \tag{8}$$

or

$$(a_i t + b_i) k t^2 + c_i \cdot k t + d_i k + \sigma x_i = 0. \tag{9}$$

Solving these equations for the ratios, we find

$$\begin{aligned} &kt^2 : -kt : k : -\sigma \\ &= (|\, bcx \,|\, t + |\, bdx \,|) : (|\, acx \,|\, t + |\, adx \,|) : |\, abx \,|\, t : U_3^* \\ &= |\, cdx \,| : (|\, adx \,|\, t + |\, bdx \,|) : (|\, acx \,|\, t + |\, bcx \,|) : U_3'. \end{aligned} \tag{10, 11}$$

As in (4), it follows from (10) that

$$\begin{aligned} -t &= (|\, bcx \,|\, t + |\, bdx \,|)/(|\, acx \,|\, t + |\, adx \,|) \\ &= (|\, acx \,|\, t + |\, adx \,|)/|\, abx \,|\, t; \end{aligned} \tag{12}$$

similarly, from (11)

$$\begin{aligned} -t &= |\, cdx \,|/(|\, adx \,|\, t + |\, bdx \,|) \\ &= (|\, adx \,|\, t + |\, bdx \,|)/(|\, acx \,|\, t + |\, bcx \,|). \end{aligned} \tag{13}$$

Multiplying and transposing in (12) and (13) yields the system of equations

$$\begin{aligned} &|\, abx \,|\, t^2 + |\, acx \,|\, t + |\, adx \,| = 0, \\ &|\, acx \,|\, t^2 + (|\, adx \,| + |\, bcx \,|) t + |\, bdx \,| = 0, \\ &|\, adx \,|\, t^2 + |\, bdx \,|\, t + |\, cdx \,| = 0. \end{aligned} \tag{14}$$

The determinant of (14) equated to zero is the equation of the R^3.

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* U_3 and U_3' need not be calculated, as they are not used in subsequent work.

In deriving the equation of the R^n whose parametric equations are

$$a_i kt^n + b_i kt^{n-1} + \cdots + \sigma x_i = 0 \qquad (i = 0, 1, 2), \tag{15}$$

it is never necessary to solve for the ratios kt^2, kt, k, σ in more than $(n+1)/2$ ways. The third equation of (14) might have been obtained from the first by interchanging a_i and d_i, b_i and c_i with the subsequent change of t to $1/t$. Obviously the second equation of (14) is invariant under this transformation. Further, if we make $n = 5$ in (15), the equation of the R^5 whose parametric equations are so obtained may be found by eliminating the powers of t from five equations. Three of these five equations

$$\begin{aligned}
&|\,abx\,|\,t^4 + |\,acx\,|\,t^3 + |\,adx\,|\,t^2 + |\,aex\,|\,t + |\,afx\,| = 0,\\
&|\,acx\,|\,t^4 + (|\,adx\,| + |\,bcx\,|)t^3 + (|\,aex\,| + |\,bdx\,|)t^2\\
&\qquad + (|\,afx\,| + |\,bex\,|)t + |\,bfx\,| = 0,\\
&|\,adx\,|\,t^4 + (|\,aex\,| + |\,bdx\,|)t^3\\
&\quad + (|\,afx\,| + |\,bex\,| + |\,cdx\,|)t^2 + (|\,bfx\,| + |\,cex\,|)t\\
&\qquad + |\,cfx\,| = 0
\end{aligned} \tag{16}$$

may be obtained by solving for the ratios in the two ways indicated by

$$\begin{aligned}
&a_i t^3 \cdot kt^2 + b_i t^3 \cdot kt + (c_i t^3 + d_i t^2 + e_i t + f_i)k + \sigma x_i = 0,\\
&(a_i t^3 + b_i t^2)kt^2 + c_i t^2 \cdot kt + (d_i t^2 + e_i t + f_i)k + \sigma x_i = 0.
\end{aligned} \tag{17}$$

Using the transformation mentioned above (which in this case consists in interchanging a_i and f_i, b_i and e_i, c_i and d_i with the subsequent change of t to $1/t$), the two remaining equations of the five may be obtained from the first two of (16). The third equation of (16) is invariant under the transformation. Obviously equations (15) can be solved for the ratios in as many ways as the coefficients of kt^2, kt, and k can be selected. It will be found convenient in actual work to use the scheme partly indicated in equations (17). That is, in the first and second solutions of equations (15) the coefficients of kt^2 and kt are $a_i t^{n-2}$, $b_i t^{n-2}$ and $a_i t^{n-2} + b_i t^{n-3}$, $c_i t^{n-3}$ respectively; according to the same scheme in the third solution the coefficients of kt^2 and kt are $a_i t^{n-2} + b_i t^{n-3} + c_i t^{n-4}$, $d_i t^{n-4}$, etc.

In (5) we have the equations of two lines different for different values of t, and the locus of their intersection is the R^2. In (14) we have three concurrent variable lines, and the locus of their intersection is the R^3. Hence in general the method consists in finding the locus of the points of concurrence of n concurrent lines subject to the condition that this point of concurrence be on the R^n.

PENNSYLVANIA STATE COLLEGE,
February, 1917.

FORD'S STUDIES ON DIVERGENT SERIES AND SUMMABILITY.

Studies on Divergent Series and Summability. By WALTER BURTON FORD. Michigan Science Series, Volume II. New York, The Macmillan Company, 1916. xi + 194 pp.

DURING the past twenty years there has been an ever increasing interest in the study of divergent series and their applications. Naturally a coexistent phenomenon has been a very large expansion in the volume of literature on this subject. An idea of the amount of this expansion may be gathered from the bibliography of Professor Ford's book, which, while not exhaustive, contains a list of some two hundred books and memoirs (principally memoirs), of which all but about twenty have appeared from 1895 on.

Thus it has become more and more difficult for one who has not followed recent work on divergent series to ascertain readily the known results in a certain branch of that field or the methods that have proved fruitful in studying certain aspects of the subject. This is alike a handicap for the experienced research worker whose investigations in other fields have naturally led to a consideration of divergent series, and to the beginner in research who feels attracted toward the subject of divergent series and wishes to orient himself rapidly in the field in order to find the avenues that may lead to new results.

To both of these classes of readers, as well as to many others, Professor Ford's admirable work will undoubtedly prove a boon. It presents in clear and concise fashion the fundamental features of each of the two grand divisions of divergent series, namely, asymptotic series and summable series, and in

certain subdivisions of each leads the reader to the present boundaries of knowledge, at the same time pointing the road to further advances. The appearance of such books, written by American mathematicians and published under the auspices of American universities, augurs well for the future development of mathematics in this country. All those who have this development at heart must feel grateful to Professor Ford and to the University of Michigan for the present contribution.

We turn now to a more detailed consideration of the book. As indicated in the previous paragraph it divides itself into two parts, one devoted to asymptotic series, the other to summable series. The first part comprises Chapters I–III, the second part Chapters IV and V.

In Chapter I certain fundamental theorems with regard to the Euler-Maclaurin sum formula* are derived, and their application to the study of asymptotic developments of functions is illustrated by a consideration of Stirling's series. This latter discussion is followed by a brief account of Poincaré's theory of asymptotic developments, which closes the chapter. The different theorems about the Euler-Maclaurin sum formula are concerned mainly with different forms for the remainder term in that formula, and are chosen from the point of view of their usefulness in obtaining asymptotic developments. It is certainly a convenience to have these important results collected in such compact and usable form.

Chapter II is devoted to the application of the general theorems of the previous chapter to the derivation of the asymptotic developments for a number of typical cases of fundamental importance. Two general classes of functions are considered, those defined by infinite products and those defined by infinite series. Beginning with rather simple examples, the writer proceeds by gradual steps to a point where he is able to treat the problem of obtaining the asymptotic development of the general integral function of order greater than zero. He also obtains a general theorem with regard to the asymptotic development of functions defined by power series whose coefficients satisfy a certain restrictive condition. The chapter as a whole forms an excellent intro-

* Professor Ford follows Barnes in using the designation Maclaurin sum formula instead of the above. However, as Euler's priority of discovery is well known, this usage seems to be lacking in historical accuracy.

duction to the literature that deals with the problem of determining the asymptotic development of a given function. The illustrative examples are well chosen with a view to giving the reader an adequate idea of the methods that have been found useful in attacking that problem.

In Chapter III a brief account is given of the progress that has been made in obtaining the asymptotic solutions of linear differential equations and linear difference equations. No demonstrations are given here; the writer contents himself with stating the principal results, giving numerous references to the literature and indicating certain important problems in the field whose solution is yet to be obtained.

In Chapter IV the reader is introduced to the second class of divergent series, namely, summable series. The writer selects six of the earlier and more widely used definitions of the sum of a divergent series and discusses certain of their properties. The definitions selected are the well known Cesàro and Hölder arithmetic mean definitions, Borel's exponential method, his integral definition and a generalization of the latter, and finally Leroy's extension of Borel's integral definition. The discussion of these definitions begins naturally with a proof of their consistency with the definition of convergence, i. e., a proof that any convergent series will be summable by the given definition to the value to which it converges. The treatment of consistency is made more compact and lucid by first proving a general lemma, by means of which the consistency of the various definitions is readily established.

The author then proceeds to define what he calls the boundary value condition. He wishes to consider as summable only series

$$\sum_{n=0}^{\infty} u_n \tag{1}$$

for which the corresponding power series

$$f(x) = \sum_{n=0}^{\infty} u_n x^n \tag{2}$$

has a radius of convergence equal to unity, and then accept only definitions of the sum of (1) for which

$$s = \lim_{x \to 1-0} f(x).$$

He has already stated in the preface his belief that some limitation of this sort is desirable and timely in order to avoid confusion and inconsistency in the study of series.

On this point the reviewer is not in agreement with Professor Ford. It is of course highly probable that many definitions that satisfy the condition of consistency will not be very useful in the study of divergent series, and it may be found desirable at some future time to exclude certain definitions that do satisfy this condition. However, the reviewer does not think the time is ripe for such exclusion and he further believes that the limitations that the author prescribes for the definition of summability are too narrow. He does not share Professor Ford's opinion, expressed in a footnote in the preface, that the present situation in the theory of divergent series is closely analogous to the one which caused Cauchy and Abel to rule out divergent series from the domain of analysis. At that time there was a decided vagueness with regard to the sense in which a divergent series could be considered as having a sum, and a lack of precision in the various attempts to define a sum. Whatever definitions have been proposed recently have been stated with all the precision of modern analysis, and, as far as the reviewer is aware, have not been chosen in a pure spirit of arbitrariness. They have been selected, either from their probable usefulness in studying certain interesting types of divergent series, or as natural outgrowths of earlier definitions or of attempts to frame general theories of divergent series.

If any limitations on the definition of summability are eventually agreed upon by workers in the field of divergent series, it seems natural to the reviewer that they should not exclude any series to which we would wish to ascribe a sum, provided we have a method of summation that gives to such a series a sum that is generally useful. The limitations suggested by Professor Ford do exclude many such series, for under these limitations no power series could be regarded as summable outside of its circle of convergence. Yet such a series can be summed within the polygon of summability by Borel's integral definition to a value which is the analytic extension of the function defined within the circle of convergence. It is surely convenient to ascribe this value to the series, and since we can obtain the value by one of the standard methods of summation, it would seem logical to include such a series in the class defined as summable.

After defining the boundary value condition, the chapter under consideration continues with a discussion of the relationship to this condition of the six definitions mentioned above. Following this, conditions are obtained under which these definitions possess certain other properties. The author considers the effect on the summability and value of a series of adding terms to or subtracting terms from the beginning of the series, the nature of the series formed by adding corresponding terms of two summable series and that formed by taking the Cauchy product. The results are summarized on page 91. In the remaining sections of the chapter the writer discusses absolutely summable series (Borel), uniform summability, and a few important properties of series summable by the Cesàro (or Hölder) definition, or by Borel's integral definition.

Chapter V deals with the application of the theory of summability to certain developments in orthogonal functions, namely to Fourier's series and some of the allied developments. In this connection the convergence of the developments, as a special case of summability, is considered as well. While the author does not in general obtain results not previously known, he makes here a distinct contribution to the subject in that he develops a general theory of summability of developments of this type, out of which the results for particular developments may be obtained as special cases. This serves to unify the whole treatment and to bring into greater prominence the essential features of the problem. The method of treatment is an extension of Dini's discussion of the convergence of developments of this type.

The chapter is divided into four parts. In the first part the author discusses the simplest case, the convergence and summability of the ordinary Fourier's series. In connection with this discussion he makes clear to the reader what are the essential steps in building up a general theory that will apply to developments of this type. In the next part he develops a number of general theorems about the representation of arbitrary functions by definite integrals, which form the basis for this general theory. In the third part he shows how Cauchy's calculus of residues can be used to advantage in applying the theorems of the previous part to the developments in question. He also discusses certain general properties of the functions appearing in the terms of the developments, which are useful in studying their convergence or summability.

In the fourth and final part of the chapter we find a discussion of three different types of developments, namely a class of important sine developments,* developments in Bessel's functions and developments in Legendre's functions. All these discussions are carried through on the basis of the author's general theory. The results, while not always the most general that have been obtained, are broad enough to show the power of the general theory. There is just one point in which results sufficiently complete for the applications have not been secured. In discussing the developments in terms of Bessel's functions, no information is obtained with regard to the convergence or summability of the series at the origin, or its uniform convergence or summability in the neighborhood of the origin. Without information of this sort we are not able to carry through some of the applications of these developments to problems in mathematical physics. It is true that most discussions of the developments in Bessel's functions leave this same point unsettled, but it is obviously a matter of much importance, and any treatment that overlooks it is incomplete in an essential point.

In connection with his discussion of the developments in Bessel's functions Professor Ford has very properly called particular attention to the fact that the first adequate discussion of the convergence of the Bessel's developments for points in the interval $0 < x \leqq 1$ was given by Dini, since the latter's priority in this matter has not received due recognition from certain writers.†

Chapter IV concludes with a discussion of the convergence and summability of the developments in Legendre's functions. The results here, though not the most general that have been

* These developments have in the successive terms of the series trigonometric functions of the form $\sin \lambda_n x$, where the λ's are the roots of the equation

$$z \cos z + p \sin z = 0.$$

† Cf. for example p. 374 of Whittaker and Watson's Modern Analysis, where the development theorem is stated and ascribed to Hobson. It is true that Hobson's result went beyond Dini's in that he considered functions having a Lebesgue integral and discussed uniform convergence. However, making allowance for a few errors in formulas, the result stated by Whittaker and Watson was fully established by Dini, and it is unfortunate that Dini's work should have been referred to by these writers in such a way as to emphasize the errors and overlook the substantial contribution.

obtained,* are complete in the sense that both convergence and summability are established under fairly broad conditions for the entire interval $(-1 \leqq x \leqq 1)$, thus including the points $x = -1$ and $x = 1$, which are exceptional in the same sense that the point $x = 0$ is for the developments in Bessel's functions.

At the beginning of this review two classes of readers to whom Professor Ford's book should prove very useful, have been mentioned. In concluding, a third class should be noted, namely those who are already actively engaged in research work in the field of divergent series. The literature in this field has grown so rapidly in such a short time that it is sometimes difficult, even for those who have made a point of following it, to locate readily a particular result. Professor Ford's references to the literature are on the whole very complete and have evidently been made the object of much care. In this way his book furnishes readier access, not only to the many valuable results it contains, but also to a still greater number of related results in the literature of divergent series.

A list of errata that have been noted is appended:

P. vii, eq. (1)	for	$\frac{B_2}{5.6}\frac{1}{x^5}$	read $\frac{B_2}{5.6}\frac{1}{x^5}$
P. 5, l. 7	"	(5)	" (6)
P. 8, l. 2	"	§ 5	" § 6
P. 10, l. 9	"	$\epsilon > h$	" $\epsilon < h$
P. 12, l. 1	"	$\sum_{x=a}^{b-h}$	" $\sum_{x=a}^{b-h}$
P. 22, footnote 27	"	$e^{-(1/z)}$	" e^{-z}
P. 38, l. 22	"	$x = +\theta_y$	" $x = y + \theta_y$
P. 50, eq. (48) and following eq.	"	$(-z)^{a-(1/2)-iy}$	" $(-z)^{a-(1/2)+iy}$
P. 52, l. 9	"	$-\pi < \varphi < \pi$	" $-2\pi < \varphi < 0$
P. 66, l. 8	insert the words *the given* before *equation*		
P. 93, fig. 6	for λ		read 2λ
P. 149, eq. 116	"	$-\frac{1}{\sqrt{2\pi z}}$	" $-\frac{i}{\sqrt{2\pi z}}$
P. 149, the eq. preceding (119)	"	$(2v-1)$ in exponents	" $(2v+1)$
P. 155, ll. 15, 22 and 33	"	$\begin{cases} t > 0 \\ t < 0 \end{cases}$	" $\begin{cases} t > \epsilon \\ t < \epsilon \end{cases}$
P. 162, last line of footnote	"	Nielson	" Nielsen
P. 184, l. 5	"	Allerdice	" Allardice

CHARLES N. MOORE.

* In the opinion of the reviewer attention should have been called to Gronwall's very exhaustive discussion of the summability of the developments in Legendre's functions (cf. *Math. Annalen*, vol. 75 (1914), p. 321) in connection with the references to the literature in this chapter.

SHORTER NOTICES.

The Human Worth of Rigorous Thinking. By CASSIUS J. KEYSER. Columbia University Press, 1916. 8vo, vii+ 314 pp. $1.75.

THESE essays and addresses of Professor Keyser are collected together in one volume as the result of fifteen years' consideration of various phases of mathematics, particularly of its significance as a branch of human knowledge. Everyone who has heard or read the different articles already knows that the author sees things with a poet's eye, and hears the strains of subtle music that emanate from this highest ether of the reason. The closing sentence of his book, and of the lecture widely delivered on mathematics, finds the deep-centered source of not only mathematics but also of science in the "joyous mood of the eternal Being" and both are considered to be a "sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds." For him "mathematics is but the ideal to which all thinking, by an inevitable process and law of the human spirit, constantly aspires." Consequently to challenge the value of mathematics as a study worthy of humanity is to challenge the worth of all thinking. Its significance to man is beyond measure, for "Transcending the flux of the sensuous universe, there exists a stable world of pure thought, a divinely ordered world of ideas, accessible to man, free from the mad dance of time, infinite and eternal." Its teaching may be so humanized that it will become a delight and a pleasure to all students, for "mathematics is precisely the ideal handling of the problems of life, and the central ideas of the science, the great concepts about which its stately doctrines have been built up, are precisely the chief ideas with which life must always deal, and which, as it tumbles and rolls about them through time and space, give it its interests and problems, and its order and rationality." The first three and the last two addresses of the collection are devoted primarily to considerations of the sort we have just mentioned.

There are four essays dealing with such fundamentals of thought as the figure and dimensions of the universe, hyperspace, infinity, and the existence of the hypercosmic. Two

others are closely related to these in that they discuss the logical reduction of mathematics and the multiple interpretations of systems of postulates and the deductions from them. In these we find the same lofty strain insisting upon the ability of the reason to find ultimate knowledge in mathematical theorems. "It is not in the world of sense, however precious it is and ineffably wonderful and beautiful, nor yet in the still finer and ampler world of imagination, but it is in the world of conception and thought that the human intellect attains its appropriate freedom—a freedom without any limitation save the necessity of being consistent." He is audacious enough to foresee an increase in our imagining powers so that even the ability to picture n-dimensional spaces may some day be gained by the race. In the hypercosmic he finds a world of pure thought "where is every type of order and manner of correlation and variety of relationship," rich in æsthetic interests, furnishing a sublimed and supersensuous art, the sure support of religious aspiration, permanent through all time. In hyperspaces he finds entities that have precisely whatever kind of existence may be attributed to the space of ordinary geometry. Whether such entities are found by the intellect or are created by it makes little difference.

There are also four essays on education, discussing the basis of a liberal education, graduate instruction, the functions of a university, and research in American universities. He takes issue with the doctrine that all subjects are equally valuable in a curriculum, pointing out that there are deep-rooted instincts in mankind which seek for certain things and for which any curriculum must provide. The study of the history of human activity and thought is a necessity, as well as the study of natural science; the study of mathematics, logic, or rigorous thinking in some form is not to be omitted, nor is that of the social character of man; but in a way surmounting them all is the study of beauty in all its forms, for without this, life would indeed be poor and depressing. Although the various topics that will respond to these needs may vary from century to century yet in some form or other they must always be present if education is really to be liberal in the highest sense. Would that everyone could appreciate the deep underlying principles set forth here, in this day of educational fads and crude radicalism! A university he finds is "the offspring and the appointed agent of the spirit of

inquiry, expression and servant of that imperious curiosity which in a measure impels all men and women, but with an urgency like destiny literally *drives* men and women of genius, to seek to know and to teach to their fellows whatsoever is worthy in all that has been discovered or thought, spoken and done in the world, and at the same time seeks to extend the empire of understanding endlessly in all directions throughout the infinite domain of the uncharted and unknown." As to research, the author believes in the three-fold organization of a university staff, the administration, the teaching staff, and the research staff. He characterizes his conceptions of the three thus: the great administrator is a man of constructive genius, the great teacher is a source of inspiration, the great investigator is a discloser of the harmonies and invariance hid beneath the surface of seeming disorder and of ceaseless change.

The delightful style of the author, permeated as it is with lofty sentiment, keen appreciation of beauty, wealth of imagery, striking illustrations, wide choice of terms, classical allusion, and warm feeling, makes the reading of these essays on rather abstract philosophical topics a pleasure as well as a profit. Every student of mathematics should read the book and catch the inspiration of enthusiasm for the divine science. "Mathematics is, in many ways, the most precious response that the human spirit has made to the call of the infinite and eternal. It is man's best revelation of the 'Deep Base of the World.'"

JAMES BYRNIE SHAW.

Four Lectures on Mathematics. (Delivered at Columbia University in 1911.) By J. HADAMARD, Member of the Institute, Professor in the Collège de France and in the Ecole Polytechnique, Lecturer in Mathematics and Mathematical Physics in Columbia University for 1911. New York, Columbia University Press, 1915. v + 52 pp.

ALL those who have had the pleasure of hearing Professor Hadamard's lectures have doubtless remarked his unusual facility in opening up wide vistas in the course of a relatively brief discussion. It is natural to expect such a facility to appear at greatest advantage in a set of lectures that are intended to be primarily inspirational, such as the above, and in this case the expectation is amply realized. By the omission of practically all technical details, Professor Hadamard has

succeeded in giving a bird's-eye view of some very extensive domains of mathematics in the course of these four lectures.

The first lecture deals with solutions of linear partial differential equations determined by boundary conditions, and the central topics are Cauchy's problem and Dirichlet's problem. The essential differences in nature of these two problems are set forth in clear and elegant fashion, and an illuminating discussion of their relationship to certain important physical problems is given. The lecture closes with a quotation from Poincaré on the service of physics to analysis in furnishing important problems whose solutions can frequently be predicted on physical grounds.

The second lecture deals with contemporary researches in differential, integral, and integro-differential equations. In discussing the solution of Dirichlet's problem by means of integral equations, the writer reminds us that the complete solution of a mathematical problem suggested by a physical problem may only be arrived at after interest in the physical problem has been lost. But at the same time he consoles us by pointing out the compensating feature that the mathematical solution may be of very great use in connection with other physical problems of greater actuality. The rest of the lecture concerns itself mainly with important investigations in differential and integro-differential equations that have been inspired by the consideration of certain physical problems. It is pointed out that under the present hypothesis of the discrete structure of matter, physical problems tend to lead to ordinary differential equations rather than to partial ones.

The third lecture is devoted to a discussion of analysis situs. The essential nature of this important branch of mathematics and its invaluable assistance in developing the theory of certain other branches, is very clearly brought out in brief compass. It is shown that in dealing with certain types of problems, analysis alone is insufficient to resolve all the difficulties without the powerful aid rendered by analysis situs. The rôle of each branch is shown in very illuminating fashion by comparison with a map of a large country given on a series of partial leaves that can only be put together by means of an "assembling table" showing the general disposition of the leaves.

The fourth and final lecture is devoted to elementary solu-

tions of partial differential equations and Green's functions. A brief but comprehensive résumé of recent investigations concerned with these two topics is given. The writer points out that the important difference between the Green's function and the elementary solution of the corresponding differential equation, corresponds to a similar difference between Cauchy's problem and Dirichlet's problem. That is to say the Green's function, like Dirichlet's problem, depends very closely on the form of a certain surface or hypersurface, whereas the elementary solution, like Cauchy's problem, does not. From this fact it is readily seen that considerations of analysis situs will play an important rôle in the study of Green's functions. Hence these functions are related to all the principal topics of the preceding lectures, and therefore, as the writer expresses it, a discussion of them forms an appropriate conclusion.

CHARLES N. MOORE.

Die elliptischen Funktionen und ihre Anwendungen. Erster Teil. By ROBERT FRICKE. Leipzig, B. G. Teubner, 1915. x+500 pp. Price 22 Marks.

THE present volume is the first of a series of three which Dr. Fricke proposes to write on the elliptic functions and their applications. It appeared in October, 1915, and is devoted to the function theoretic and analytic bases of the theory of elliptic functions. One would naturally expect that a treatise on elliptic functions from the pen of Dr. Fricke would follow the lines of thought developed by Klein and his students thirty-odd years ago. Consequently, on turning the pages of the present volume, one is not surprised to be reminded again and again of modes of thought, of formulas, and of geometric diagrams made familiar through the Klein-Fricke Modulfunktionen. Dr. Fricke refers to this when he writes in the preface: "That I should adhere in the main to the methods of presentation, the use of the invariant theory, geometric representation, and so forth, which more than thirty years of close scientific companionship with my teacher and friend F. Klein have made my own, I may regard as self-evident."

The introduction, consisting of 105 pages, is devoted to an exposition of theorems concerning analytic functions of a single complex variable. This material is made to lead up to a statement of the basic problems of the theory of elliptic func-

tions; namely, a study of the algebraic functions and their integrals on a Riemann surface whose deficiency is $p = 1$, and a classification of these functions into essentially different stages (Stufentheorie). The introduction closes with an application of the Fuchs' theory to linear differential equations of the second order, in particular to the hypergeometric differential equation and hypergeometric series.

The remainder of the volume is divided into two parts (Abschnitte), of which the first, consisting of five chapters, is devoted to the theory of elliptic functions of the first stage (Stufe). This, as we know, is the "Weierstrass theory" with its characteristic rational invariants g_2, g_3 and the corresponding normal form of the integral of first kind

$$u = \int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

The discussion contains the usual materials with the invariant theory and the group theory always in the foregound. The first part closes with a discussion of the "differential equation of the periods" and the related Schwarz differential expression.

The second part, consisting of three chapters, opens with a discussion of irrational invariants of the algebraic form $f(z)$ of the fourth degree and we soon reach the characteristic irrational invariant λ (anharmonic ratio) of the second stage, and the irrational invariant μ (octahedral irrationality), characteristic of the fourth stage, together with the corresponding normal forms of the integral of first kind

$$u = \int \frac{dz}{\sqrt{z(1-z)(1-\lambda z)}},$$

$$u = \int \frac{dz}{\sqrt{(1-z^2)(1-\mu^4 z^2)}}.$$

The first of these is the Riemann normal integral, and the second resembles very closely the Legendre-Jacobi normal form, but our author prefers to call it the Abel normal form, since Abel used it in his Recherches sur les Fonctions elliptiques. The Legendre-Jacobi normal form differs from the Abel normal form in having k^2 in place of μ^4. Moreover the Jacobi elliptic functions belong to the second stage in the Stufentheorie and not to the fourth.

The explicit statement of Klein's Stufentheorie occurs in the second chapter. This theory depends upon the group of linear substitutions $\Gamma^{(u,\,\omega)}$, that is,

$$(1)\qquad \begin{aligned} u' &= u + m_1\omega_1 + m_2\omega_2, \\ \omega_1' &= \alpha\omega_1 + \beta\omega_2, \\ \omega_2' &= \gamma\omega_1 + \delta\omega_2, \end{aligned}$$

where m_1, m_2, α, β, γ, δ are integers and $\alpha\delta - \beta\gamma = 1$, and its congruence subgroups; the principal congruence subgroup of nth stage being defined by the congruences

$$(2)\quad m_1 \equiv 0,\ \ m_2 \equiv 0,\ \ \alpha \equiv \delta \equiv 1,\ \ \beta \equiv \gamma \equiv 0 \pmod{n}.$$

The group $\Gamma^{(u)}$; that is,

$$u' = u + m_1\omega_1 + m_2\omega_2,$$

and the modular group $\Gamma^{(\omega)}$ are considered separately. The principal congruence subgroup of nth stage of $\Gamma^{(u)}$ is defined by the first two congruences of (2), and the principal congruence subgroup of nth stage of $\Gamma^{(\omega)}$ is defined by the last four congruences of (2). The theory of the modular group and its principal congruence subgroups is contained in the Modulfunktionen (1890). The definition of the discontinuity domains for the principal congruence subgroups of nth stage of $\Gamma^{(u)}$ and of $\Gamma^{(\omega)}$ follow, and we are led, finally, to the following definition of an elliptic function of nth stage.

An elliptic function of nth stage is a uniform homogeneous function $\psi(u/\omega_1, \omega_2)$ of integral dimensions d of the three arguments u, ω_1, ω_2 which is invariant (covariant) under the substitutions of the principal congruence subgroup of nth stage of $\Gamma^{(u,\,\omega)}$; for fixed values of ω_1, ω_2, it is an analytic function of u, free from essential singularities in the discontinuity domain of the principal congruence subgroup of nth stage of $\Gamma^{(u)}$ and is defined for the entire u-plane except for $u = \infty$; and for a fixed value of u/ω_2

$$\omega_2^{-d}\psi(u/\omega_1, \omega_2) = \psi\left(\frac{u}{\omega_2}\middle|\, \omega, 1\right) \quad \left[\omega = \frac{\omega_1}{\omega_2}\right]$$

is an analytic function of ω, free from essential singularities within the discontinuity domain of the principal congruence subgroup of nth stage of the modular group $\Gamma^{(\omega)}$, and is defined for the positive ω-half plane.

Nearly a quarter of a century ago Professor Klein expressed the hope that he might sometime be able to treat the whole theory of elliptic functions from this point of view,* and the present text, when completed, may be regarded as in a sense a fulfillment of that hope.

The Jacobi functions, $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$, now appear as elliptic functions of the second stage and the remainder of the volume is devoted to their properties, their relations to the Weierstrass functions, and to the properties of the allied theta functions.

Dr. Fricke's style is well known the world over and needs no comment. The text is free from errors, due credit being given by the author to Fräulein Dr. H. Petersen for careful proof-reading. The book appeared from the Teubner press "in spite of all the difficulties of the time, without unusual interruptions." In conclusion we may express the hope that nothing will interfere with the publication of the two remaining volumes.

L. WAYLAND DOWLING.

Problems in the Calculus, with Formulas and Suggestions. By DAVID D. LEIB, Ph.D. Boston, Ginn and Company, 1915. xii+224 pp. Price $1.00.

THE first impression conveyed by the title of this book, that it may be a collection of problems of the sort discussed by Professor Archibald in the BULLETIN of June, 1914, is corrected by the first few words of the preface. The book is said to be "the outgrowth of lists of problems prepared by the author to supplement the textbook," and in it we find "a supplementary list of workable problems on any topic ordinarily included in a general course in the calculus."

Preceding each set of exercises is a brief statement of the formulas and methods applicable thereto, with warnings against some common errors. In fact it seems to the reviewer that too careful a classification and too definite directions are given to permit, much less encourage, the development of initiative, versatility, and flexibility on the part of the student. When he has worked through a reasonable number of examples in his text-book, where properly enough most of them are carefully classified, the best additional preparation for the

* Evanston Colloquium (Aug. 28–Sept. 9, 1893). Lecture X.

unassorted difficulties he will meet in physics, mechanics, or elsewhere would seem to be furnished by examples in which the method of attack is left for him to discover. This is especially true in the case of formal integration; many students can "turn the crank" when they know what crank to turn, but are helpless when it comes to selecting a suitable method for themselves. This book of problems would be more helpful to them if the author had written each problem on formal integration for instance on a separate piece of paper and had secured the cooperation of a playful kitten in completing the arrangement.

The intentional omission of the answers to a large number of examples should assist the development of students, but one would hardly make that claim for the inaccuracy which is threatened by the author's statement that "no one has formally verified the answers." The reviewer has not done so in many cases, but would call attention to errors in parts (*b*), (*e*), (*f*), (*g*), (*i*) of example 3, exercise XCIX and in examples 3 and 15 in exercise CII. The last-named example is the approximate integration of

$$\int_0^{\frac{1}{2}\pi} \frac{dx}{1+\cos x}.$$

In the first place, it hardly seems worth while to use approximate rules for more than one or two examples that can be done easily by direct integration; or at least it is not worth while to preserve many such problems and answers in a book. In the second place, it is hardly worth while to employ the methods of approximate integration unless it is done accurately enough to give better results than inspection, counting squares, or guessing in any other form. The answer given for this problem, 1.057 as compared with the correct 1, is surely beyond permissible limits of error. It apparently results from a hasty use of the trapezoidal rule, which inspection shows to be inexact in this case. The same unwise use of this rule is suggested in example 3 of the same exercise, to find by four ordinates the value of $\int_0^1 \sqrt{1+x^4}\,dx$. If this is treated as an area, the slope of the curve at the upper point is $\sqrt{2}$, and the slope of the upper boundary of the last trapezoid is less than 1; obviously the trapezoidal rule will be inaccurate. The answer

given, 1.07, seems to be the result of hasty work; accurate use of the trapezoidal rule yields 1.103, as compared with the exact value, 1.0894, obtained by the aid of Γ functions and verified by Simpson's rule with five ordinates. The error due to the trapezoidal rule is nominally only 1.3 per cent., but inspection shows the area to be a unit square surmounted by a small area, and for this latter even an accurate use of the rule suggested results in an error of about 15 per cent. Such approximate integration seems unsatisfactory to the reviewer, although it must be admitted that other books appear to sanction it.

The points open to criticism—the too orderly arrangement, inaccuracies in certain answers, and undesirable features in a few problems—are flaws which limit rather than destroy the usefulness of the book. It is certain that many teachers will find it convenient to put in the hands of students for supplementary work, and more will find it a very satisfactory source from which to draw, for classroom work or tests, problems which will be new to their students.

R. W. BURGESS.

Vom periodischen Dezimalbruch zur Zahlentheorie. By ALFRED LEMAN. Leipzig, Teubner, 1916. iv+59 pp. Price 80 Pfennige.

THE aim of this little book is to present the main properties of periodic decimals as material for a concrete introduction to the more elementary topics of the theory of numbers. The concept congruence is introduced on page 16 and Fermat's theorem is presented, illustrated and proved on pages 21–23, —each topic being a natural sequel to concrete questions and results concerning periodic decimals.

Periodic fractions to bases other than 10 are treated briefly in the final chapter 8, although on page 51 this part of the theory is said to be treated in chapters 8 and 9.

There is a list of about 15 papers, including most of the earliest ones. There is no mention of the early MS. by Leibniz; Henry Goodwyn's tables, 1816–23; the papers by Poselger, 1827; Bredow, 1834; Midy, 1836; Catalan, Thibault, and Sornin in *Nouvelles Ann. Math.*, 1842, 1843, 1846; Desmarest's text, 1852; Hudson's excellent paper in Oxford, *Cambridge and Dublin Messenger of Mathematics*, 1864, pages 1–6—not to cite various earlier papers and a hundred later ones. The topic is not exhaustively treated as claimed.

In the notes on page 54, the statements concerning perfect numbers and the largest known prime are not up to date.

However, the author has succeeded in his aim to write an attractive account of the main properties of periodic decimals and to use that topic as a concrete means to acquaint the immature reader with some important theorems of the theory of numbers and to arouse his curiosity to pursue the theory further.

L. E. DICKSON.

Compendio de Álgebra de Abenbéder. Textó árabe, traducción y estudio por JOSÉ A. SÁNCHEZ PÉREZ. Madrid, 1916. (Junta para Ampliación de Estudios e Investigaciones científicas, Centro de Estudios históricos.)

THE translator, who has an enthusiastic hope that some day an adequate history of mathematics in Spain will be written, sees in this manuscript a contribution of material for such a book. But the translation is of value for another reason: it makes accessible to the mathematical world in general another of the works compiled by the Mussulmans between the eighth and fifteenth centuries. Sr. Pérez begins his introduction with a brief account of certain phases of the history of mathematics in his own country and closes it by thirty pages of discussion of questions relating to the contents of this document and its authorship and date.

The Compendio de Álgebra is contained in manuscript 936 of the Escurial library (Arabic section). There are forty-six folios in Arabic characters of Spanish type. The date, as given in the document, is the year 744 since the Hegira. Abenbéder understands that the object of algebra is the solution of equations. He explains his subject in the form of ordinary discourse, without employing the notations of algebra. The work is divided into two parts—the theoretical and the practical. In the former, the first six "questions" treat the six forms of equations given by Al-Khowarizmi. Some of the particular equations used are those met with so frequently in these early texts: for example, (in our notation) $x^2 + 10x = 39$ and $x^2 + 21 = 10x$. There are also six "chapters" explaining the fundamental operations with the square roots of numbers and six others dealing with the rules of signs and with the squares and cubes of the unknowns.

The practical part of the book starts out with six problems illustrating the six types of equations. Five of these might well be placed in the next chapter, which contains eleven problems "concerning ten." In each of these sixteen exercises, the number ten is to be divided into two parts according to to some specified condition: for example, the product of the parts divided by their difference equals five and one fourth. Then follow seven collections containing forty-eight examples in all. Half of these are "problems of the squares"; such as, one more than one third of a square multiplied by one more than one fourth of the square equals twenty. Another group of five questions about soldiers involves arithmetical progressions. There are also "problems of commerce," "problems of the gifts," and others concerning our old friends the couriers, who have, by various methods of travel, been pursuing one another so industriously through the pages of algebra textbooks for centuries. After a formal statement that the end of the book of "chéber y almocábala" has been reached, there are appended a problem involving an arithmetic progression and also three meager rules for the solution of the three types of the complete quadratic.

E. B. COWLEY.

First Year Mathematics. By GEORGE W. EVANS and JOHN A. MARSH. New York, Chas. E. Merrill Company, 1916.

As its name indicates, this work is intended for use in the first year of the high-school course, and is a correlation of elementary algebra, plane geometry and the fundamental ideas of coordinate and locus.

In Chapter 1, simple equations in one unknown are introduced as a means of abbreviating arithmetical processes, with applications to ratio, linear and angular measurement, valuation problems, angles and angle relations and circular measurement. It may be noted that the term "stripe" is introduced in this chapter to denote a pair of parallel lines. Also, that it is pointed out in detail that precision of measurement is indicated by the number of significant figures in the result rather than by the number of decimal places. In explaining negative quantities the historical method is followed by assigning to the negative the primitive idea of a shortage to be made up, or caused to disappear, by the addition of a quantity sufficient to cover the shortage. The

word shortage is the term actually used in the text, with a note to the effect that the process of making up a shortage is the real significance of the Arab word algebra, of which the English equivalent is "restoration."

Chapter 2 takes up approximate computations, including multiplication, division, and square root, with practical rules for fixing the decimal point and determining the accuracy of the result.

Chapter 3 relates to the measurement of areas of plane figures, including the triangle, trapezoid, and circle, and the evaluation of simple formulas.

Chapter 4 is devoted to the solution of fractional equations and certain standard types of problems, notably that of numbers formed by certain digits, the problem of relative velocities, and the ancient problem of ability and time. The problems in this chapter seem to be either irrelevant or of merely historical interest, and might well be replaced by problems of modern application and significance.

Chapter 5 considers simple transformations of algebraic expressions such as transposition and the use of parentheses, and the multiplication and division of polynomials. The algebraic negative is here again explained as a shortage, such for example as arises when a merchant is unable to fill an order because of insufficient stock on hand.

Chapter 6 explains the distinction between an equation of condition and an algebraic identity, and also takes up the factoring of quadratic expressions.

Chapter 7 is devoted to the solution of quadratic equations. An important feature of this discussion is the careful and complete explanation given of the practical meaning of like and unlike roots, of negative and imaginary roots, of meaningless answers, etc.

In Chapter 8 the elementary theorems of plane geometry are treated, including the properties of similar triangles, the ratio properties of triangles, and the Pythagorean theorem, the latter also being treated by the aid of the trigonometric tangent.

Chapter 9 explains the fundamental idea of coordinate and locus, and illustrates by plotting data giving rise to a linear graph. To illustrate the relation of converse statements, such as "If an equation is of the first degree its locus is a straight line," and its converse "If a locus is a straight line

its equation is of the first degree," the pupil is required to give the converse of certain other statements, such as "If my straw hat is ruined a horse has stepped on it."

In Chapter 10 simultaneous linear equations are solved by elimination and by combination.

In Chapter 11 simultaneous equations, including quadratics, are solved by the method of substitution, and also graphically by the intersection of their loci.

Chapter 12 consists of 30 pages of supplementary problems for review.

As a brief summary of the treatment, it may be said that formal definitions are reduced to a minimum, and every effort is made to appeal to the common sense of the pupil. The text is so arranged as to give the pupil a clear idea of the meaning and purpose of algebra and geometry, rather than to set forth a logical and conventional system of mathematical doctrine. This is primarily the ideal to be attained in the coordination of elementary mathematics, and as this idea seems to have been clearly the guiding principle in the selection and arrangement of material in this case, it is interesting to note the vitality and unity it gives to the text. Such a one volume text necessarily covers a limited field, but within its limits it is, with a few exceptions, thoroughly modern in its spirit and aims, as well as eminently teachable.

S. E. Slocum.

Second-Year Mathematics for Secondary Schools. 2d edition. By E. R. Breslich. University of Chicago Press, 1916.

This book is one of the numerous recent attempts to correlate elementary algebra, geometry, and trigonometry for purposes of instruction. The idea of correlation is of course one of great possibilities, but to be successfully realized it must be based on some central and unifying principle, such, for instance, as the function concept advocated by Klein. A careful examination of the present work fails to reveal any such principle of selection or arrangement, and leaves the impression of a haphazard collection of unrelated topics.

In order that the book may be judged on its own merits and not by the opinion of the reviewer, the following brief summary of its contents is given.

Chapter 1 is a tabular statement of the geometric theorems

and constructions included in the preceding volume on first year mathematics.

Chapter 2 is a brief explanation of logical statement, fallacies, and methods of proof, which could hardly be appreciated by pupils at this point, before having had sufficient practice to distinguish between the methods described and the difficulties involved.

Chapter 3 relates to algebra, including simultaneous linear equations, elimination by combination and by substitution, and by the intersection of the graphs of their loci. This arrangement is meritorious, but the treatment is entirely too brief and is accompanied by very few problems.

Chapter 4 takes up geometry, beginning with the properties of quadrilaterals and parallelograms. This is interrupted by four quadratic equations to be solved algebraically, followed by three problems on loci, a page and a half on prisms and a page and a half on dihedral angles. This chapter shows plainly the lack of coordination which characterizes the book.

Chapter 5 treats of proportional line segments, supplemented by properties of transversals intercepted by a system of parallel planes, and includes photographs of two railway bridges.

Chapter 6 begins with the theory of proportion, is interrupted by three pages on the factoring of polynomials, returns to proportion, and ends by showing the relation of direct and inverse variation to proportion.

Chapter 7 treats of similar plane figures, but digresses to explain the graphical method for finding mechanically the quotient of two arithmetical numbers. The objection to such an irrelevant digression is that it violates the principle of correlation which aims at concentrating the pupils' thought instead of diverting it.

Chapter 8 starts out by defining the projection of one line on another, then proceeds to the simplification of radicals, passing next to a discussion of the theorem of Pythagoras with historical notes and a picture of Fermat. The formula for the solution of quadratic equations is then given, and the chapter concludes by generalizing the theorem of Pythagoras. How this arrangement facilitates instruction is not fully apparent.

Chapter 9 is mainly on trigonometry. The sine, cosine and tangent are defined, and brief tables of these functions given, with applications to the solution of plane triangles,

followed by the derivation of two trigonometric identities, after which comes the algebraic and graphical solutions of a quadratic with a linear equation.

Chapter 10 relates to theorems on the circle, with historical notes and a photograph of Dryburgh Abbey.

Chapters 11 and 12 are on geometry, discussing the measurement of angles by circular arcs, and proportional line segments in circles.

Chapter 13 is on algebra and considers fractional equations including trigonometric identities.

Chapter 14 swings back to geometry, taking up geometric inequalities mainly applied to line segments.

Chapter 15 begins with dihedral angles and concludes with certain theorems on the sphere, enlivened by a photograph of the cathedral of S. Maria del Fiore.

Chapter 16 treats of geometric loci, including among other things definitions of the circumcenter, incenter, excenter, orthocenter, the nine point circle and the Eulerian line.

Chapter 17 is on the relation of a circle to its inscribed and circumscribed circles, and includes the determination of the value of π. This is illustrated by a photograph of Gauss, with historical notes, and photos of two ancient German buildings.

Chapter 18 begins with a few theorems on plane areas, passing from this to the solution of literal equations in one and two unknowns, then back again to the discussion of certain theorems on areas of triangles, and ending with the factoring of polynomials.

Chapter 19 concludes the work with a discussion of areas of polygons, illustrated with photographs of two peculiar buildings, which are not named but apparently represent what is called the Chicago style of architecture.

The modern demand for economy of time in education will eventually lead to correlation of mathematics in which the central idea will be to cultivate in the pupil the habit of mathematical thought and exact expression, to give him equal facility in the application of algebraic and geometric methods, and to make the subject matter vital by modern applications and interpretation. The outline given above seems to indicate that the opportunity to accomplish these results has been almost entirely overlooked in the present work, as the arrangement has no advantage over similar material selected at random from standard texts. S. E. Slocum.

A Course on the Solution of Spherical Triangles for the Mathematical Laboratory. By HERBERT BELL, M.A., B.Sc., Assistant in Natural Philosophy in the University of Edinburgh. London, G. Bell and Sons, Limited, 1915. viii + 66 pp.

THIS little book is No. 5 of the Edinburgh Mathematical Tracts of which Nos. 1, 2, and 4 are also manuals for the mathematical laboratory. It presents an account of various methods, numerical and graphical, of solution of spherical triangles.

The author keeps his object clearly in mind, goes directly at it, maintains a good proportion of parts in definition, exposition, and illustration, and the result is a practical syllabus of the subject in brief space. The value of this manual is considerably enhanced for the inexperienced computer by the foresight of the author in going into details to exhibit and illustrate methods and devices which arise from experience which the beginner has not had, and are not immediately evident. On the other hand the reader is not confused with long expositions where they are not needed, and when he has once been over the contents and has acquired some experience in computing he will find the book what it was intended to be, a useful and practical manual.

The first chapter deals with the use of logarithmic tables, including addition and subtraction logarithms, and questions of precision. Seven place tables are used in the examples. The second chapter is an exposition of the general spherical triangle, the cases which arise on the basis of D'Ocagne's classification, and the fundamental formulas, including those for solution by means of auxiliary angles. The third and fourth chapters are devoted to the numerical solution of the right-angled and the general spherical triangle, respectively.

Chapter V treats certain special applications to navigation and astronomy, including great circle sailing, finding the longitude at sea, reducing an angle to the horizon, and conversion of star coordinates. In Chapter VI graphical methods of solution are set forth at some length. Here are given Monge's method, the Emerson-Langley construction, nomograms, including the analemma, Chauvenet's solver, and straight-line nomograms.

A. M. KENYON.

United States Life Tables: 1910. Prepared in the Division of Vital Statistics of the Bureau of the Census under the supervision of JAMES W. GLOVER. Washington, D. C., Bureau of the Census, 1916. 65 pp.

THIS work includes twenty-five life tables derived from twenty-five classes of persons of the population of the original registration states. The statistical data for these tables are obtained from the estimated population within these states as of July 1, 1910, and from the corresponding deaths in the calendar years 1909, 1910, and 1911. Osculatory interpolation retaining fifth differences was employed in distributing into yearly age groups the numbers given in quinquennial age groups. The first table in the book (pages 16–17) is derived from the entire population of the original registration states. The remaining twenty-four tables are obtained by classifications of this population into male and female, white and negro, rural and urban, native born and foreign born, and by preparing distinct tables for the following states: Indiana, Massachusetts, Michigan, New Jersey, and New York.

There has been urgent need of an application of mathematico-statistical methods to derive reliable results on mortality among the general population, and among various classes of that population. Both from an examination of the present work and through correspondence with Professor Glover, the reviewer is much impressed with the thoroughgoing methods employed in deriving these tables. The results are based on the part of the population of the United States on which we have, in general, more reliable mortality data than on other parts of the country. It is a very important feature of this work that separate tables of infant mortality are prepared, which show the mortality by age intervals of one month for the first year of life, and that the method used in preparing the tables for the first five years of life is the very commendable one used in constructing the German life tables for the decennium 1891–1900. An explanation of the meanings of various columns of the tables is given, with concrete examples, which will make the tables easily understood even by persons who are not particularly familiar with the language in which mortality rates, populations living at given ages, and expectations of life are expressed. One may easily draw from these tables rather striking information in regard to comparative mortality rates of different classes of persons. For

example, of the classes considered, the negro males have the highest mortality, and the white females from the rural population have the lowest mortality. The tables give as the complete expectation of life at birth for the male negro 34.05 years and for the rural female 57.35 years. The average death rate per thousand of the total population of the former is 29.37 while that of the latter is 17.44. As another striking illustration, we note that white males in cities have a complete expectation at birth of 47.32 years and an average death rate per 1,000 of 21.13, while the rural white males have a corresponding expectation of 55.06 years, and an average death rate per thousand of 18.16.

These tables may well become standard for certain civil purposes, such as the valuation of life estates, where there does not exist the selection of lives such as is involved in the data on which life insurance tables are based.

H. L. RIETZ.

NOTES.

THE March number (series 2, volume 18, number 3) of the *Annals of Mathematics* contains the following papers: "Symmetric functions formed by systems of elements of a finite algebra and their connection with Fermat's quotient and Bernoulli's numbers," by H. S. VANDIVER; "The generalized Lagrange indeterminate congruence for a composite ideal modulus," by H. S. VANDIVER; "On the congruence $cx^\lambda + 1 \equiv dy^\lambda$ in a Galois field," by H. H. MITCHELL; "On the geodesics and geodesic circles on a developable surface," by W. C. GRAUSTEIN; "Note on representations of the partial sum of a Fourier's series," by DUNHAM JACKSON; "Acknowledgment," by FRANK IRWIN; "Certain general properties of functions," by HENRY BLUMBERG.

Under an arrangement by which the Mathematical Association of America will contribute to its financial support, the *Annals* will enlarge its annual volume by 100 pages, which will be devoted to expository and historical papers. The subscription price, beginning with volume 19, will be $3.00, with a reduction of one half to members of the Association.

THE November, 1916, number (volume 2, number 11) of the *Proceedings of the National Academy of Sciences* contains

the following mathematical papers: "A set of independent postulates for cyclic order," by E. V. HUNTINGTON; "Certain general properties of functions," by HENRY BLUMBERG. The February and March, 1917, numbers (volume 3, numbers 2 and 3) contain: "Natural and isogonal families of curves on a surface," by JOSEPH LIPKA; "Some problems of Diophantine approximation: the series $\Sigma e(\lambda_n)$ and the distribution of the points $(\lambda_n \alpha)$," by G. H. HARDY and G. E. LITTLEWOOD; "A note on the fitting of parabolas," by J. R. MINER; "The complete enumeration of triad systems in 15 elements," by F. N. COLE, L. D. CUMMINGS, and H. S. WHITE.

AT the meeting of the London mathematical society held on January 18 the following papers were read: By G. H. HARDY and S. RAMANUJAN: "Asymptotic formulas in combinatory analysis"; by M. J. M. HILL: "The singular solutions of ordinary differential equations of the first order"; by H. BATEMAN: "The nature of a moving electric charge and its lines of electric force"; by L. J. ROGERS: "The expansion of the variables of a hypergeometric equation in terms of the ratio of two solutions"; by H. J. PRIESTLEY: "A problem in the theory of diffraction."

THE Swiss mathematical society held its regular summer meeting at Schuls (Engadine) August 6–9, 1916. The following papers were presented: By L. CRELIER: "The power of a line"; by O. SPIESS: "Closure problems for convex curves"; by C. CAILLER: "On imaginary ruled geometry"; by W. H. YOUNG: "Multiple integrals and Fourier series"; by GRACE C. YOUNG: "On the curves of Cellérier and Weierstrass"; by F. RUDIO: "Report on the new edition of Euler"; by M. GROSSMANN: "The completion of the general theory of relativity"; by H. WEYL: "The problem of analysis situs"; by L. G. DU PASQUIER: "On generalized arithmetic"; by G. PÓLYA: "A counterpart of the Liouville approximation theorem in the theory of differential equations"; by O. BERLINER: "On a projective natural geometry"; by K. MERZ: "Historical note on the Steiner surface"; by W. H. YOUNG and GRACE C. YOUNG: "The structure of functions of several variables"; by O. BLOCH: "On the geometry of the Gauss number plane."

AT the annual meeting of the Paris academy of sciences held on December 21, the following prizes were awarded in pure and applied mathematics:

As no satisfactory memoir was received on the problem set in 1915, the Grand prize was not awarded. But a prize of 2,000 francs was given to Professor N. E. NÖRLUND, of the University of Lund, for his contributions to the theory of linear equations in finite differences. The de Parville prize (1500 francs) was awarded to Professor L. TORRES, of Madrid, for his calculating machine for solving algebraic equations and other mechanisms invented by him. The prize in history and philosophy of the sciences (1,000 francs) was given to Professor J. BENSAUDE, of Lisbon, for his book on nautical astronomy in Portugal during the great exploration period. The Francoeur prize (1,000 francs) was awarded to the late Professor L. COUTURAT, of the Collège de France, for his mathematical works. The Poncelet prize (2,000 francs) was awarded to Professor C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, for his mathematical works. The Lalande, Valz, and Janssen prizes in astronomy were divided among J. E. COGGIA, G. BOCCARDI, CH. FABRY, H. BUISSON, and H. BOURGET.

The following prizes are offered under the usual conditions, the awards to be made in December, 1917: Poncelet prize (2,000 francs) for a memoir in pure mathematics; Francoeur prize (1,000 francs) for the best essay in pure or applied mathematics; Montyon prize (700 francs) for the best practical application of mathematics; Fourneyron prize (1,000 francs) for the most meritorious investigation of ball bearings. The problem set for 1916 is also repeated: the most important improvement of motors used in aviation. The Boileau prize (1,300 francs) for theoretical and experimental progress in hydraulics; de Parville prize (1,500 francs) for a contribution to mechanics; Lalande prize (540 francs) and Valz prize (460 francs) for researches in astronomy; Grand prize (3,000 francs) for the best essay on the problem: "To improve in an important point the study of the successive powers of the same substitution, as the exponent of the power increases indefinitely;" Le Conte prize (50,000 francs) for work in mathematics (one eighth may be used for encouragement, and seven eighths in one or more prizes for results). The general prizes include, for recognition of mathematical accomplishment without more detailed specification, the following: Houllevigue prize (5,000 francs), Wilde prize (4,000 francs) and Jérome Ponti prize (3,500 francs).

THE biennial Ackermann-Teubner prize (1,000 M) has been awarded to Professor E. ZERMELO, of the University of Zürich, for his investigations of the theory of sets, particularly for his memoir of 1907 on well ordered sets.

THE following university courses in mathematics are announced:

COLLÈGE DE FRANCE: (December 1, 1916–March 25, 1917).—By Professor G. HUMBERT: Abelian functions of two variables, two hours.—By Professor J. HADAMARD: Partial differential equations and the problem of Cauchy, two hours.—By Dr. M. BRILLOUIN (from January 2): Internal structure of the earth according to geodesy and seismology, two hours.—By Dr. L. LANGEVIN: Principle of relativity and the theory of gravitation, two hours.

COLUMBIA UNIVERSITY (summer session, July 9–August 17).—By Professor JAMES MACLAY: Fundamental mathematical concepts, five hours; Theory of geometric constructions, five hours; Theory of numbers, five hours.—By Professor EDWARD KASNER: Differential geometry, five hours; Theory of functions of a complex variable, five hours.—By Professor W. B. FITE: Projective geometry, five hours; Higher algebra (Galois theory of equations), five hours.

CORNELL UNIVERSITY (summer session, July 9–August 17).—By Professor V. SNYDER: Foundations of elementary mathematics, five hours; Ruler and compasses, five hours.—By Professor C. F. CRAIG: Higher analysis, five hours.—By Professor F. W. OWENS: Projective geometry, five hours. Professors SHARPE, CARVER, GILLESPIE, HURWITZ and Drs. MCKELVEY and SILVERMAN will conduct a mathematical conference weekly.

UNIVERSITY OF PENNSYLVANIA (summer session, July 9–August 18).—By Professsor G. H. HALLETT: Higher calculus, five hours.—By Professor R. L. MOORE: Foundations of geometry, five hours.—By Dr. F. W. BEAL: Continuous groups, five hours.

UNIVERSITY OF PENNSYLVANIA (academic year 1917–18).—By Professor E. S. CRAWLEY: Higher plane curves, two

hours.—By Professor G. E. FISHER: Functions of a complex variable, two hours.—By Professor I. J. SCHWATT: Infinite series and products, two hours.—By Professor G. H. HALLETT: Finite groups, two hours.—By Professor F. H. SAFFORD: Partial differential equations, two hours.—By Professor M. J. BABB: Theory of numbers, two hours.—By Professor G. G. CHAMBERS: Synthetic projective geometry, two hours.—By Professor O. E. GLENN: Calculus of variations (second semester), two hours.—By Professor H. H. MITCHELL: Algebraic numbers, two hours.—By Professor R. L. MOORE: Foundations of mathematics, two hours.—By Dr. F. W. BEAL: Differential geometry, two hours.

PROFESSOR C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, has been elected a foreign associate of the Royal society of Naples.

DR. A. TERRACINI has been appointed instructor in analytic geometry in the University of Turin.

PROFESSOR A. M. HARDING, of the University of Arkansas, has been promoted to a full professorship of mathematics.

AT Harvard University the two Benjamin Peirce instructorships have been filled for the coming year by the reappointment of Dr. W. L. HART and the appointment of Dr. T. A. PIERCE. Mr. B. H. BROWN and Mr. J. L. WALSH have been appointed instructors in mathematics. Dr. L. R. FORD, of the University of Edinburgh, has been appointed instructor in actuarial mathematics.

DR. G. R. CLEMENTS, of the University of Wisconsin, has been appointed instructor in mathematics in the U. S. Naval Academy.

PROFESSOR WILLIAM BEEBE, of Yale University, died March 11 at the age of sixty-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDION (I.). La cuadratura del circulo y de la eliptica. La Pay, R. C. Piuto, 1916. 4to.

BOOLE (G.). Collected logical works. Volume 2: Laws of thought (1854). London, Open Court, 1916. 8vo. 16+448 pp. 15s.

DUNKEL (O.). See GOURSAT (E.).

GOURSAT (E.). A course in mathematical analysis. Differential equations, being Part 2 of Volume 2. Translated by E. R. Hedrick and O. Dunkel. Boston, Ginn, 1917. 8vo. 8+300 pp. $2.75

HANCOCK (H.). Elliptic integrals (Mathematical Monographs, No. 18.) New York, Wiley, 1917. 8vo. 104 pp. Cloth. $1.25

HEDRICK (E. R.). See GOURSAT (E.).

HUNTINGTON (E. V.). The continuum and other types of serial order. 2d edition. Cambridge, Mass., Harvard University Press, 1917. 8vo. 8+82 pp. Cloth. $1.00

INDEX du répertoire bibliographique des sciences mathématiques. 3e édition. Amsterdam, Delsman, 1916. 115 pp.

JOHNSON (G.). The arithmetical philosophy of Nicomachus of Gerasa. (Diss., Pennsylvania.) Lancaster, Pa., 1916. 49 pp.

LEHMER (D. N.). Elementary course in synthetic projective geometry. Boston, Ginn, 1917. 12mo. 13+123 pp. $0.96

LIGHT (G. H.). The dependence of the topography of envelopes of systems of extremals on curvature. (Diss., Yale.) Lancaster, Pa., 1917. 4to. 32 pp.

MACROBERT (T. M.). Functions of a complex variable. London, Macmillan, 1917. 14+295 pp. 12s.

MARCH (H. W.) and WOLFF (H. C.). Calculus. (Modern Mathematical Texts, edited by C. S. Slichter.) New York, McGraw-Hill, 1917. 8vo. 16+360 pp. $2.00

MITCHELL (B. E.). Complex conics and their real representation. (Diss., Columbia.) Lancaster, Pa., 1917. 8vo. 4+44 pp.

RICE INSTITUTE. The book of the opening of the Rice Institute. Houston, Rice Institute, 1917. 8vo. Vol. 1. Pp. 14+1–264. Vol. 2. Pp. 265–680. Vol. 3. Pp. 681–1100.

SEDGWICK (W. T.) and TYLER (H. W.). A short history of science. New York, Macmillan, 1917. $2.50

SLICHTER (C. S.). See MARCH (H. W.).

SOPRA la insussistenza del numero π. Palermo, Calogero Sciarrino, 1916. 8vo.

TYLER (H. W.). See SEDGWICK (W. T.).

WOLFF (H. C.). See MARCH (H. W.).

II. ELEMENTARY MATHEMATICS.

ANDREWS (P. L.). Test papers in mathematics: designed to cover the last year's work of preparation for the public schools and for the Royal naval college, Osborne; with an introduction by E. F. Johns. Portsmouth, Gieves (John Hogg), 1916. 8vo. 48 pp. 1s. 6d.

ATWOOD (G. E.). See MYERS (G. W.).

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